Optimisation of conic portfolios
Conic portfolio theory

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Part I

Introduction to portfolio theory
Market as the centre of the model

- $A_i$: an asset.
- $Z_i$: the cash flow of $A_i$ between two dates $(t_0, T)$.
- $X_i$: the value of the asset at the horizon date $T$.
- A probability space $(\Omega, \mathcal{F}, P)$.

The non-arbitrage hypothesis

$$E_q[X_i] = (1 + r)X_i^{(0)}$$

The market accepts to...

- Buy: $Z_i = X_i - (1 + r)w$, with $w \leq X_i^{(0)}$.
- Sell: $Z_i = (1 + r)w - X_i$, with $w \geq X_i^{(0)}$. 
Acceptability sets

- The following condition is satisfied:

\[ E_q[Z_i] \geq 0. \]

Acceptability set

\[ \mathcal{A} = \{ Z \mid E_q[Z] \geq 0 \} \]

- Given a set of probability measures \( \mathcal{M} \).

Generalized acceptability set

\[ \mathcal{A} = \{ Z \mid E_q[Z] \geq 0 \ \forall q \in \mathcal{M} \} \]
Bid and ask prices

- The market agrees to buy $A_i$ for $b$ or sell it for $a$ if
  $$X_i - b(1 + r) \in A, \quad a(1 + r) - X_i \in A$$

- Which means that for all $q \in \mathcal{M}$
  $$E_q[X_i] - b(1 + r) \geq 0,$$
  $$a(1 + r) - E_q[X_i] \geq 0.$$

Bid and Ask formulas

$$b(X) = \frac{1}{1 + r} \inf_{q \in \mathcal{M}} E_q[X],$$

$$a(X) = \frac{1}{1 + r} \sup_{q \in \mathcal{M}} E_q[X].$$
Bid price

- The function that we want to maximise is:

\[ b(X) = \inf_{q \in \mathcal{M}} E_q[X]. \]

- Assuming comonotone additivity and law invariance:

\[ b(X) = \int_{\mathbb{R}} x \, d\Psi(F_X(x)). \]
Coherent risk measures

A *coherent risk measure* is a function of the form

$$\rho(X) = - \inf_{q \in \mathcal{M}} E_q[X] = -b(X).$$

**Remark:** To simplify the notation one can think everything in terms of the bid price $b(X)$.

**Definitions**

- Set of supporting kernels:

  $$\mathcal{M} = \{ q \in \mathcal{P} \mid E_q[X] \geq b(x) \ \forall q \in L^\infty(\Omega) \}.$$

- Set of extreme measures $Q^*(X)$ defined as:

  $$E_q[X] = b(X) \ \forall q \in Q^*(X).$$
Coherent risk measures

Acceptability set

\[ A = \{ X \in L^\infty(\Omega) \mid b(X) \geq 0 \}. \]
Indexes of acceptability

- Acceptability index: coherent risk measure satisfying
  - Quasi-concavity.
  - Monotonicity.
  - Scale invariance.
  - Fatou property.

- Coherent risk measure for an increasingly set \((\mathcal{M}_x)_{x \in \mathbb{R}^+}\): 
  \[
  b_x(X) = \inf_{q \in \mathcal{M}_x} E_q[X].
  \]

- Index of acceptability 
  \[
  \alpha(X) = \sup\{x \mid b_x(X) \geq 0\}.
  \]

- Remark: with this properties the acceptability set is a convex cone.
TVaR measure

**TVaR coherent risk measure**

\[
TVaR_\lambda(X) = -\inf_{q \in \mathcal{M}_\lambda} E_q[X].
\]

**TVaR equivalent definition**

\[
TVaR_\lambda(X) = E_q[X | X \leq x_\lambda(X)].
\]
WVaR index of acceptability

- A generalisation of TVAR.
- It is the average of $\text{TVAR}_\lambda$ with different risk levels $\lambda$ weighted by a probability measure $\mu$.

\[
\text{WVAR}_\mu(X) = \int_0^1 \text{TVAR}_\lambda(X) \mu(d\lambda).
\]

- We can find an alternative definition that looks like the bid price:

\[
\text{WVAR}_\mu(X) = -\int_{\mathbb{R}} y d(\psi_\mu(F_X(y))).
\]
Stress level

- We define a one parameter family of concave functions $\Psi^\gamma$.
- $\gamma$: stress level.

**WVaR acceptability index**

$$AIW(X) = \sup\{\gamma \mid b_\gamma(X) = \int_{\mathbb{R}} y \, d(\Psi^\gamma(F_X(y))) \geq 0\}.$$**

- **Conclusion:** Stress level is equivalent to portfolio risk (or acceptability).
MinMaxVaR y Wang Transform

**MinMaxVaR**

\[ \Psi^\gamma(u) = 1 - \left( 1 - u^{\frac{1}{1+\gamma}} \right)^{1+\gamma}. \]

**Wang Transform**

\[ \Psi_\Phi^\gamma(u) = \Phi(\Phi^{-1}(u) + \gamma). \]
Part II

Calibration and computations
Bid price discretisation by Madan

- The portfolio return over an investment time horizon is:

\[ R_p = \sum_{i=0}^{N} a_i (e^{x_i} - 1). \]

- **Goal**: find the optimal \( a_i \) to maximise the return, or more precisely the bid of it.

\[
 b(R_p) = \sum_{m=1}^{M} \sum_{i=1}^{N} a_i (e^{x_i,m} - 1) \left( \psi^\gamma \left( \frac{m}{M} \right) - \psi^\gamma \left( \frac{m-1}{M} \right) \right). 
\]
Problem: this inequality appears because we are taking the assets separated so a correlation should be considered.

Problem: this presents a difficulty on the computation and we are assuming a particular form on the distribution.
(i) For each of the $N$ stocks estimate the bid price ($b'$) and the ask price ($a'$) from market data.
   
   - bid price ($b'$): the minimum price of the previous 63 days.
   - ask price ($a'$): the maximum price of the previous 63 days.

(ii) Relativize each of the previous estimated quantities to the average price ($\bar{x}$) of the previous 63 days:

\[
b = \frac{b'}{\bar{x}}, \quad a = \frac{a'}{\bar{x}}.
\]
Calibration algorithm for the stress level

1.- For each calibration date $t$: take the average of the bid $b$ and ask $a$ along the stocks.

$$b_t = \sum_{i=1}^{N} a_i b_t^{(i)}, \quad b_t = \sum_{i=1}^{N} b_t^{(i)}.$$

2.- We estimate the stress level with least squares minimisation:

$$\gamma_t = \arg \min_{\gamma} \left( b_t - \hat{b}_t(\gamma) \right)^2 + (a_t - \hat{a}_t(\gamma))^2, \quad (1)$$

where

$$\hat{b}_t(\gamma) = b_t(X, \gamma) = \sum_{m=1}^{M} x_m \left( \psi^{\gamma} \left( \frac{m}{M} \right) - \psi^{\gamma} \left( \frac{m-1}{M} \right) \right).$$
Part III

Optimisation problems
Optimisation problem

- Let $R_p = \sum_{i=1}^{N} a_i R_i$ be a portfolio.

General Optimisation problem

\[
\text{find: } \max_{a_i} b(R_p), \\
\text{subject to: } \sum_{i=1}^{N} a_i = 1.
\]

Centred returns

\[
R_i = \mu_i + R_i^\varepsilon \Rightarrow b(R_i) = \mu_i + b(R_i^\varepsilon).
\]

\[
b(R_p) = \bar{a} \cdot \bar{\mu} + b(R_p^\varepsilon).
\]
Long-only Optimisation problem

find: \( \max_{\mathbf{a}} \mathbf{a} \cdot \mathbf{\mu} + b(R_p^\varepsilon) \),

subject to: \( \sum_{i=1}^{N} a_i = 1. \)
Long-short portfolios

- Using centred returns as before:

$$b(R_p) = \mathbf{a} \cdot \mathbf{\mu} - \tilde{c}(\mathbf{a}).$$

Long-short optimisation problem

find: \( \min_{\mathbf{a}_i} \tilde{c}(\mathbf{a}) \),

subject to: \( \mathbf{a} \cdot \mathbf{1} = 1 \),

\( \mathbf{a} \cdot \mathbf{\mu} = \mu_p \).

- **Idea**: efficient frontier.