Optimisation of conic portfolios

Conic portfolio theory

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31 de marzo de 2016

Part I

Introduction to portfolio theory

Market as the centre of the model

- A_i: an asset.
- Z_i: the cash flow of A_i between two dates (t_0, T).
- X_i: the value of the asset at the horizon date T.
- A probability space $(\Omega, \mathcal{F}, P)$.

The non-arbitrage hypotesis

$$E_q[X_i] = (1+r)X_i^{(0)}$$

The market accepts to...

- Buy: $Z_i = X_i - (1+r)w$, with $w \leq X_i^{(0)}$.
- Sell: $Z_i = (1+r)w - X_i$, with $w \geq X_i^{(0)}$.

Acceptability sets

- The following condition is satisfied:

$$E_q[Z_i] \geq 0.$$

Acceptability set

$$\mathcal{A} = \{Z \mid E_q[Z] \geq 0\}$$

- Given a set of probability measures $\mathcal{M}$.

Generalized acceptability set

$$\mathcal{A} = \{Z \mid E_q[Z] \geq 0 \quad \forall q \in \mathcal{M}\}.$$

Bid and ask prices

- The market agrees to buy A_i for b or sell it for a if

$$X_i - b(1+r) \in \mathcal{A}, \quad a(1+r) - X_i \in \mathcal{A}$$

- Which means that for all $q \in \mathcal{M}$

$$\begin{aligned} E_q[X_i] - b(1+r) &\geq 0, \\ a(1+r) - E_q[X_i] &\geq 0. \end{aligned}$$

Bid and Ask formulas

$$\begin{aligned} b(X) &= \frac{1}{1+r} \inf_{q \in \mathcal{M}} E_q[X], \\ a(X) &= \frac{1}{1+r} \sup_{q \in \mathcal{M}} E_q[X]. \end{aligned}$$

Bid price

- The function that we want to maximise is:

$$b(X) = \inf_{q \in \mathcal{M}} E_q[X].$$

- Assuming comonotone additivity and law invariance:

$$b(X) = \int_{\mathbb{R}} x \,\mathrm{d}\Psi(F_X(x)).$$

Coherent risk measures

- A *coherent risk measure* is a function of the form

$$\rho(X) = -\inf_{q \in \mathcal{M}} E_q[X] = -b(X).$$

- **Remark:** To simplify the notation one can think everything in terms of the bid price $b(X)$.

Definitions

- Set of supporting kernels:

$$\mathcal{M} = \{q \in \mathcal{P} \mid E_q[X] \geq b(x) \;\; \forall q \in \mathrm{L}^{\infty}(\Omega)\}.$$

- Set of extreme measures $Q^*(X)$ defined as:

$$E_q[X] = b(X) \;\; \forall q \in Q^*(X).$$

Coherent risk measures

Acceptability set

$$\mathcal{A} = \{X \in L^{\infty}(\Omega) \mid b(X) \geq 0\}.$$

Indexes of acceptability

- Acceptability index: coherent risk measure satisfying
  - Quasi-concavity.
  - Monotonicity.
  - Scale invariance.
  - Fatou property.
- Coherent risk measure for an increasingly set $(\mathcal{M}_x)_{x\in\mathbb{R}_+}$:

$$b_x(X) = \inf_{q\in\mathcal{M}_x} E_q[X].$$

Index of acceptability

$$\alpha(X) = \sup\{x \mid b_x(X) \geq 0\}.$$

- **Remark:** with this properties the acceptability set is a convex cone.

TVaR measure

TVaR coherent risk measure

$$\mathrm{TVaR}_\lambda(X) = -\inf_{q\in\mathcal{M}_\lambda} E_q[X].$$

TVaR equivalent definition

$$\mathrm{TVaR}_\lambda(X) = E_q[X|X \leq x_\lambda(X)].$$

WVaR index of acceptability

- A generalisation of TVAR.
- It is the average of TVAR_λ with different risk levels λ weighted by a probability measure μ.

WVaR

$$\mathrm{WVAR}_\mu(X) = \int_0^1 \mathrm{TVAR}_\lambda(X)\mu(\mathrm{d}\lambda).$$

- We can find an alternative definition that looks like the bid price:

WVaR alternative definition

$$\mathrm{WVAR}_\mu(X) = -\int_{\mathbb{R}} y\,\mathrm{d}(\Psi_\mu(F_X(y))).$$

Stress level

- We define a one parameter family of concave functions Ψ^γ.
- γ: stress level.

WVaR acceptability index

$$\mathrm{AIW}(X) = \sup\{\gamma \mid b_\gamma(X) = \int_{\mathbb{R}} y \, \mathrm{d}(\Psi^\gamma(F_X(y))) \geq 0\}.$$

- **Conclusion:** Stress level is equivalent to portfolio risk (or acceptability).

MinMaxVaR y Wang Transform

MinMaxVaR

$$\Psi^{\gamma}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}.$$

Wang Transform

$$\Psi^{\gamma}_{\Phi}(u) = \Phi(\Phi^{-1}(u) + \gamma).$$

Part II

Calibration and computations

Bid price discretisation by Madan

- The portfolio return over an investment time horitzon is:

$$R_p = \sum_{i=0}^{N} a_i(e^{X_i} - 1).$$

- **Goal**: find the optimal a_i to maximise the return, or more precisely the bid of it.

Bid price discretisation

$$b(R_p) = \sum_{m=1}^{M} \sum_{i=1}^{N} a_i \left(e^{X_{i,m}} - 1\right) \left(\Psi^{\gamma}\left(\frac{m}{M}\right) - \Psi^{\gamma}\left(\frac{m-1}{M}\right)\right).$$

Another bid price discretisation

Bid price discretisation

$$b(R_p) = \inf_{q\in\mathcal{M}} E_q[R_p] \leq \inf_{\widetilde{q}\in\mathcal{M}} \sum_{i=1}^{N} a_i \left(\frac{E_{\widetilde{q}_i}[X_i^{(f)}]}{X_i^{(0)}} - 1 \right)$$

- **Problem**: this inequality appears because we are taking the assets separated so a correlation should be considered.
- **Problem**: this presents a difficulty on the computation and we are assuing a particular form on the distribution.

Bid price discretisation

$$b(R_p) = \inf_{\theta_1,\dots,\theta_N} \sum_{i=1}^{N} a_i \left(\frac{E_{q^{\theta_i}}[X_i^{(f)}]}{X_i^{(0)}} - 1 \right).$$

Calibration algorithm for the stress level

(*i*) For each of the N stocks estimate the bid price (b') and the ask price (a') from market data.

- bid price (b'): the minimum price of the previous 63 days.
- ask price (a'): the maximum price of the previous 63 days.

(*ii*) Relativize each of the previous estimated quantities to the average price ($\bar{x}$) of the previous 63 days:

$$b = \frac{b'}{\bar{x}}, \qquad a = \frac{a'}{\bar{x}}.$$

Calibration algorithm for the stress level

1.- For each calibration date t: take the average of the bid b and ask a along the stocks.

$$b_t = \sum_{i=1}^{N} a_i b_t^{(i)}, \qquad b_t = \sum_{i=1}^{N} b_t^{(i)}.$$

2.- We estimate the stress level with least squares minimisation:

$$\gamma_t = \arg\min_{\gamma} \left(b_t - \widehat{b}_t(\gamma)\right)^2 + (a_t - \widehat{a}_t(\gamma))^2, \tag{1}$$

where

$$\widehat{b}_t(\gamma) = b_t(X, \gamma) = \sum_{m=1}^{M} x_m \left(\Psi^{\gamma}\left(\frac{m}{M}\right) - \Psi^{\gamma}\left(\frac{m-1}{M}\right)\right).$$

Part III

Optimisation problems

Optimisation problem

- Let $R_p = \sum_{i=1}^{N} a_i R_i$ be a portfolio.

General Optimisation problem

$$\text{find:} \quad \max_{a_i} \; b(R_p),$$

$$\text{subject to:} \quad \sum_{i=1}^{N} a_i = 1.$$

Centred returns

$$R_i = \mu_i + R_i^{\varepsilon} \;\Rightarrow\; b(R_i) = \mu_i + b(R_i^{\varepsilon}).$$

$$b(R_p) = \vec{a} \cdot \vec{\mu} + b(R_p^{\varepsilon}).$$

Long-only portfolios

Long-only Optimisation problem

$$\text{find:} \quad \max_{a_i} \; \vec{a} \cdot \vec{\mu} + b(R_p^{\varepsilon}),$$

$$\text{subject to:} \quad \sum_{i=1}^{N} a_i = 1.$$

Long-short portfolios

- Using centred returns as before:

$$b(R_p) = \vec{a} \cdot \vec{\mu} - \tilde{c}(\vec{a}).$$

Long-short optimisation problem

$$\begin{aligned} \text{find:} \quad & \min_{a_i} \ \tilde{c}(\vec{a}), \\ \text{subject to:} \quad & \vec{a} \cdot \mathbf{1} = 1, \\ & \vec{a} \cdot \vec{\mu} = \mu_p. \end{aligned}$$

- **Idea**: efficient frontier.