Introduction to Limit Order Books

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Limit Order Book

Description

1. A Limit Order Book ("LOB") is a trading method used by most exchanges globally. It is a transparent system that matches customer orders (e.g. bids and offers) on a 'price time priority' basis.
2. Customers can see market depth. Both bid orders and ask orders for various sizes and prices are public.

Limit Order Book

Advantages

- transparent
- low cost
- everybody can trade with everybody else
- trading is anonymous

Limit Order Book

Definition: Order

Definition

A *sell order* $x = (p_x, \omega_x, t_x)$ submitted at time t_x with price p_x and size $\omega_x > 0$ is a commitment to *sell* up to $|\omega_x|$ units of the traded asset at a price no less than p_x.

Definition

A *buy order* $x = (p_x, \omega_x, t_x)$ submitted at time t_x with price p_x and size $\omega_x < 0$ is a commitment to *buy* up to $|\omega_x|$ units of the traded asset at a price no greater than p_x.

Definition

The *tick-size* π the smallest permissible price. All orders must arrive with an accuracy of π.

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Trade-matching algorithm

- When a buy (sell) order x is submitted, a matching-algorithm checks whether is is possible to match x to some other previously submitted sell (buy) order. If so, the matching occurs immediately. If not, x stays in the system, waiting to be matched or to be cancelled.
- An order which is immediately matched is called *market order*.
- An orders which is not immediately matched is called *limit order*.
- Usually price-time priority is used for matching.

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Limit Order Book

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A LOB $\mathscr{L}(t)$ is the set of all limit orders in a market at time t.

Remark

$\mathscr{L}(t)$ is a càdlàg process. (right continuous with left limits). It holds for $x \in \mathscr{L}(t_x), x \notin \lim_{t \uparrow t_x} \mathscr{L}(t)$.

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Limit Order Book

Definition bid/ask-side

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A LOB $\mathscr{L}(t)$ can be partitioned into the set of buy orders $\mathscr{B}(t)$ for which $\omega_x < 0$, and into the set of sell orders $\mathscr{A}(t)$ for which $\omega_x > 0$.

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The set of buy orders are also called *bid-side*. The set of sell orders are also called *ask-side*.

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Limit Order Book

Definition best bid/ask

Definitions

The *best bid price* at time t is the highest stated price among the buy orders at time t,

$$b(t) := \max_{x \in \mathscr{B}(t)} p_x.$$

The *best ask price* at time t is the lowest stated price among the sell orders at time t,

$$a(t) := \min_{x \in \mathscr{A}(t)} p_x.$$

The couple *best bid price* and *best ask price* is also called *top of the book.*

Limit Order Book

Definition spread, mid-price, volume

Definition

The *bid-ask spread* at time t is $s(t) := a(t) - b(t)$.

Definition

The *mid-price* at time t is $m(t) := [a(t) + b(t)]/2$.

Definition

The *bid-side volume* or *depth* available at price p and time t is

$$v_b(p,t) := \sum_{\{x \in \mathscr{B}(t) | p_x = p\}} \omega_x.$$

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Video

- VIDEO

LOB data

- LOB data offer unusually rich, detailed and high-quality historic data.
- The data provides testing ground for theories e.g. about statistical regularities.
- Investigations of LOB come from:
    - Economics
    - Physics
    - Statistics
    - Mathematics
    - Psychology

Modelling Approaches

Perfect Rationality versus Zero Intelligence

- Economists usually focus on the behaviour of individual traders. (perfect rationality). Traders are rational and maximize their personal utility.
- Statistics/ physics community assume orders to be governed by stochastic processes. (zero intelligence).
  - Parameters can be estimated by historic data.
  - Statistical output of such models can be compared to real data.

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Key unresolved Problems

- Understand statistical regularities.
- Provide a model that is capable of simultaneously reproduce all so far known stylized facts. (Such model does not yet exists!).
- Understand recent data.
- Algorithmic trading:
  - very few empirical studies so far.
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Model Framework

Define the n'th model (bid side only)

- Let $n \in \mathbb{N}_0$. For each n define a model describing the LOB.
  - Δx^n is the tick size of the n'th model. We could analysis $\Delta x^n \to 0, n \to \infty$.
  - Δt^n is the inter-arrival time between two orders. Time t is discrete, $t \in \Delta t^n \cdot \mathbb{N}$.
  - B_t^n is a stochastic process describing the best bid price at time t.
  - T_t^n is a stochastic process describing the volume at the best bid price.
  - $V_t^{n,i}$ is a stochastic process describing the relative volume i ticks away from the best bid price at time t. $i = 0, 1, 2, \ldots$ The absolute volume i ticks in the book at time t is $T_t^n \cdot v_t^{n,i}$.

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Orders

- There are two types of orders:
  - passive orders
  - active orders
- The random variable $\gamma_t^n \in \{a, p\}$ decides for each order arriving at time t whether is a passive or an active order.

Passive Orders

Passive orders do not lead prices to change. We distinguish two different event types:

- Limit order placement or cancellation at the top of the book by the random factor $\alpha_t^n \in (0, \infty)$.
- Limit order placement or cancellation in the book by the random factor $\beta_t^n \in [0, \infty)$.

Let ω_t^n be a non-negative random variable describing the place, where limit order placement or cancellation happens.

Passive Orders
Dynamics

The dynamics of passive orders at the bid-side are as follows:

$$B_t^n - B_{t-}^n = 0$$

$$T_t^n - T_{t-}^n = \mathbf{1}_{\omega_t^n < \Delta x^n} \left(\alpha_t^n - 1 \right) T_{t-}^n$$

$$\begin{aligned} V_t^{n,k} - V_{t-}^{n,k} &= \mathbf{1}_{\omega_t^n < \Delta x^n} \left(\frac{1}{\alpha_t^n} - 1 \right) V_{t-}^{n,k} \\ &\quad + \mathbf{1}_{\omega_t^n \geq \Delta x^n} \mathbf{1}_{k = \lceil \omega_t^n / \Delta x^n \rceil} \left(\beta_t^n - 1 \right) V_{t-}^{n,k}, \quad k \in \mathbb{N} \end{aligned}$$

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Active Orders

Active orders lead prices to change. There are two different event types:

- A limit buy-order is placed into the spread. The volume placed is described by the positive random variable ξ_t^n. For simplicity it is assumed that this happens exactly one tick above the (previously) best bid price.
- A market order arrives. For simplicity it is assumed that such order exactly wipes out the volume at the best bid price.

Let $\eta_t \in \{0,1\}$ be a random variable.

- The event $\eta_t = 0$ corresponds to the event "limit order into the spread".
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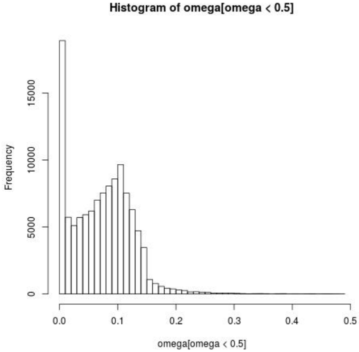
Simulations

- **Simulations**

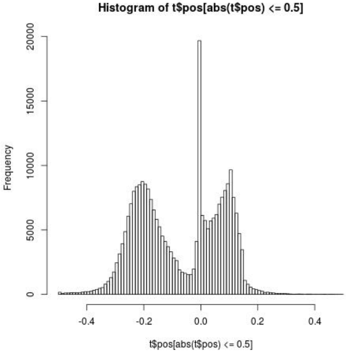
Omega

Histogram of Omega for Amazon.com on 2012-06-21. Price at start of trading: 223.82 USD.



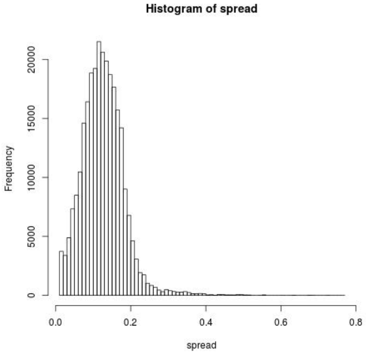
Positions

The diagram shows a histogram of the distance between the best bid price and the position where placement/ cancellation of orders takes place. (Amazon.com on 2012-06-21).



Spread

The diagram shows a histogram of the spread. (Amazon.com on 2012-06-21).



Positions

The diagram shows a histogram of the distance between the best bid price and the position where placement/ cancellation of orders takes place. (Microsoft Corporation on 2012-06-21). Price at start of trading: 30.95 USD.