Introduction to Limit Order Books

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February 4, 2016
A Limit Order Book ("LOB") is a trading method used by most exchanges globally. It is a transparent system that matches customer orders (e.g. bids and offers) on a 'price time priority' basis.

Customers can see market depth. Both bid orders and ask orders for various sizes and prices are public.
Limit Order Book

Advantages

- transparent
- low cost
- everybody can trade with everybody else
- trading is anonymous
Limit Order Book

Definition: Order

Definition

A **sell order** $x = (p_x, \omega_x, t_x)$ submitted at time $t_x$ with price $p_x$ and size $\omega_x > 0$ is a commitment to *sell* up to $|\omega_x|$ units of the traded asset at a price no **less** than $p_x$.

Definition

A **buy order** $x = (p_x, \omega_x, t_x)$ submitted at time $t_x$ with price $p_x$ and size $\omega_x < 0$ is a commitment to *buy* up to $|\omega_x|$ units of the traded asset at a price no **greater** than $p_x$.

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The **tick-size** $\pi$ the smallest permissible price. All orders must arrive with an accuracy of $\pi$. 
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The *tick-size* $\pi$ the smallest permissible price. All orders must arrive with an accuracy of $\pi$. 
When a buy (sell) order $x$ is submitted, a matching-algorithm checks whether it is possible to match $x$ to some other previously submitted sell (buy) order. If so, the matching occurs immediately. If not, $x$ stays in the system, waiting to be matched or to be cancelled.

- An order which is immediately matched is called *market order*.
- An orders which is not immediately matched is called *limit order*.
- Usually price-time priority is used for matching.
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A LOB $\mathcal{L}(t)$ is the set of all limit orders in a market at time $t$.

**Remark**

$\mathcal{L}(t)$ is a càdlàg process. (right continuous with left limits). It holds for $x \in \mathcal{L}(t_x), x \not\in \lim_{t \uparrow t_x} \mathcal{L}(t)$.


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Limit Order Book
Definition bid/ask-side

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A LOB $\mathcal{L}(t)$ can be partitioned into the set of buy orders $\mathbb{B}(t)$ for which $\omega_x < 0$, and into the set of sell orders $\mathbb{A}(t)$ for which $\omega_x > 0$.

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The set of buy orders are also called bid-side. The set of sell orders are also called ask-side.
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Definitions

The *best bid price* at time $t$ is the highest stated price among the buy orders at time $t$,

$$ b(t) := \max_{x \in \mathcal{B}(t)} p_x. $$

The *best ask price* at time $t$ is the lowest stated price among the sell orders at time $t$,

$$ a(t) := \min_{x \in \mathcal{A}(t)} p_x. $$

The couple *best bid price* and *best ask price* is also called *top of the book*. 
Definition

The bid-ask spread at time $t$ is $s(t) := a(t) - b(t)$.

Definition

The mid-price at time $t$ is $m(t) := [a(t) + b(t)]/2$.

Definition

The bid-side volume or depth available at price $p$ and time $t$ is

$$v_b(p, t) := \sum_{\{x \in B(t) | p_x = p\}} \omega_x.$$
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Definition spread, mid-price, volume

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Basic Notion
State of the Art (briefly)
Discrete Limit Order Book Model

Video

VIDEO
LOB data

- LOB data offer unusually rich, detailed and high-quality historic data.
- The data provides testing ground for theories e.g. about statistical regularities.
- Investigations of LOB come from:
  - Economics
  - Physics
  - Statistics
  - Mathematics
  - Psychology
Economists usually focus on the behaviour of individual traders. (perfect rationality). Traders are rational and maximize their personal utility.

Statistics/physics community assume orders to be governed by stochastic processes. (zero intelligence).

- Parameters can be estimated by historic data.
- Statistical output of such models can be compared to real data.
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Key unresolved Problems

- Understand statistical regularities.

- Provide a model that is capable of simultaneously reproduce all so far known stylized facts. (Such model does not yet exists!).

- Understand recent data.

- Algorithmic trading:
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Model Framework
Define the n’th model (bid side only)

- Let $n \in \mathbb{N}_0$. For each $n$ define a model describing the LOB.
  - $\Delta x^n$ is the tick size of the n’th model. We could analysis $\Delta x^n \to 0, n \to \infty$.
  - $\Delta t^n$ is the inter-arrival time between two orders. Time $t$ is discrete, $t \in \Delta t^n \cdot \mathbb{N}$.
  - $B^n_t$ is a stochastic process describing the best bid price at time $t$.
  - $T^n_t$ is a stochastic process describing the volume at the best bid price.
  - $V^{n,i}_t$ is a stochastic process describing the relative volume $i$ ticks away from the best bid price at time $t$. $i = 0, 1, 2, ...$ The absolute volume $i$ ticks in the book at time $t$ is $T^n_t \cdot V^{n,i}_t$. 
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Orders

- There are two types of orders:
  - passive orders
  - active orders

- The random variable $\gamma_t \in \{a, p\}$ decides for each order arriving at time $t$ whether is a passive or an active order.
Passive Orders

Passive orders do not lead prices to change. We distinguish two different event types:

- Limit order placement or cancellation at the top of the book by the random factor $\alpha_n^t \in (0, \infty)$.
- Limit order placement or cancellation in the book by the random factor $\beta_n^t \in [0, \infty)$.

Let $\omega_n^t$ be a non-negative random variable describing the place, where limit order placement or cancellation happens.
The dynamics of passive orders at the bid-side are as follows:

\[ B^n_t - B^n_{t-} = 0 \]

\[ T^n_t - T^n_{t-} = 1_{\omega^n_t < \Delta x^n} (\alpha^n_t - 1) T^n_{t-} \]

\[ V^{n,k}_t - V^{n,k}_{t-} = 1_{\omega^n_t < \Delta x^n} \left( \frac{1}{\alpha^n_t} - 1 \right) V^{n,k}_{t-} \]

\[ + 1_{\omega^n_t \geq \Delta x^n} 1_k = \left\lceil \omega^n_t / \Delta x^n \right\rceil (\beta^n_t - 1) V^{n,k}_{t-}, \quad k \in \mathbb{N} \]
The dynamics of passive orders at the bid-side are as follows:

\[ B_t^n - B_{t-}^n = 0 \]

\[ T_t^n - T_{t-}^n = 1_{\omega_t^n < \Delta x^n} (\alpha_t^n - 1) T_{t-}^n \]

\[ V_{t, k}^n - V_{t-}^{n, k} = 1_{\omega_t^n < \Delta x^n} \left( \frac{1}{\alpha_t^n} - 1 \right) V_{t-}^{n, k} \]

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Passive Orders
Dynamics

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Active orders lead prices to change. There are two different event types:

- A limit buy-order is placed into the spread. The volume placed is described by the positive random variable $\xi^n_t$. For simplicity it is assumed that this happens exactly one tick above the (previously) best bid price.

- A market order arrives. For simplicity it is assumed that such order exactly wipes out the volume at the best bid price.

Let $\eta_t \in \{0, 1\}$ be a random variable.

- The event $\eta_t = 0$ corresponds to the event ”limit order into the spread”.

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Simulations
Histogram of Omega for Amazon.com on 2012-06-21. Price at start of trading: 223.82 USD.
The diagram shows a histogram of the distance between the best bid price and the position where placement/cancellation of orders takes place. (Amazon.com on 2012-06-21).
The diagram shows a histogram of the spread. (Amazon.com on 2012-06-21).
The diagram shows a histogram of the distance between the best bid price and the position where placement/cancellation of orders takes place. (Microsoft Corporation on 2012-06-21). Price at start of trading: 30.95 USD.